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JULY 1963 AIAA JOURNAL VOL. 1, NO. 7

Optimum Design of Truss-Core Sandwich Cylinders Under Axial Compression

Gerald A. Cohen*

Aeronutronic Division, Ford Motor Company, Newport Beach, Calif.

Sufficient conditions are determined for which equality of the critical stresses is necessary for minimum weight design of compression structures with two instability modes. It is shown that these conditions are satisfied for single and double truss-core sandwich cylinders under axial compression if the sandwich depth is free (i.e., not determined by other considerations, such as wall resistance to meteoroid penetration or heat transfer). Charts are presented which determine minimum weight designs of single and double truss-core sandwich cylinders if the sandwich depth is free. The optimization problem with the sandwich depth given is discussed, but no numerical work has been done to generate design charts for this case.

Nomenclature

A_c = cross-sectional area of core per unit of circumferential width
A_f = cross-sectional area of face per unit of circumferential width
b_c = width of core elements
b_f = width of face elements
E = elastic modulus
h = sandwich thickness measured between middle surfaces of face elements
I_c = cross-sectional moment of inertia of core per unit of circumferential width
I = cross-sectional moment of inertia of face per unit of circumferential width
k_g = general buckling coefficient
k_l = local buckling coefficient
k_{pq} = sequence of functions whose minimum (with respect to integral values of p and q) is k_g
L = length of cylinder
m = index denoting single truss core ($m = 1$) or double truss core ($m = 2$)
n = number of independent geometric variables required to define cross section
q = load per unit width (load intensity)
R = radius of cylinder
s = shear deformation reduction factor
t = thickness of thin-wall cylinder
$\bar{t}$ = cross-sectional area per unit width (effective thickness)
t_c = thickness of core elements
t_f = thickness of face elements
x_i = independent geometric variables that define cross section
$Z, \beta, \zeta, \Lambda, \xi, \psi$ = functions defined in text

Presented at the IAS 31st Annual Meeting, New York, January 21–23, 1963; revision received May 20, 1963. This work was performed under Aeronautical Systems Division Contract AF 33(616)-7775.

* Staff Member, Engineering Analysis Department.

η = dimensionless sandwich thickness
θ = angle between core and face element
λ = Lagrange multiplier
μ = Poisson's ratio
σ_1,σ_2 = critical stress functions for two instability modes
σ_g = critical stress for general buckling
σ_l = critical stress for local buckling
τ = thickness function
$\bar{\tau}$ = dimensionless effective thickness
τ_f = dimensionless face thickness
φ = constraint function

I. Introduction

In previous studies[1–3] of minimum weight design of compression structures, it generally has been assumed that optimum proportions result when the possible forms of buckling occur simultaneously. Simple examples may be proposed which show that minimum weight may occur when the critical stresses are *not* equal. In order to clarify this situation, a set of conditions is determined for uniformly loaded compression structures with two instability modes that are sufficient to insure that equality of the critical stresses is a necessary condition for minimum weight. This theory then is applied to single and double truss-core sandwich cylinders.

II. Statement of Problem

The basic problem considered is that of designing a cross section with minimum effective thickness $\bar{t}$ to carry a given compressive load intensity q. In general, for a given class of cross sections there are n independent geometric variables $x_i > 0 (i = 1,2, \ldots n)$ which define the cross section. All other pertinent quantities, such as load intensity, elastic modulus, or other geometric dimensions of the structure, are considered to be known input parameters. In general, the critical stresses σ_1 and σ_2 for the two instability modes also are functions of x_i. Thus the problem is, given the positive functions $\bar{t}(x_i)$, $\sigma_1(x_i)$, and $\sigma_2(x_i)$, to minimize $\bar{t}$ with respect to the x_i subject to the constraint

$$\bar{t} = q/\min(\sigma_1,\sigma_2)$$

Since q is an input parameter, it is equivalent to minimize the function $\tau = \bar{t}/q$ subject to the constraint

$$\tau = 1/\min(\sigma_1,\sigma_2) \tag{1}$$

III. Discussion of the Minimization Problem

The surface S defined by

$$\sigma_1(x_i) = \sigma_2(x_i) \tag{2}$$

divides the x_i space $(x_i > 0)$ into two open regions denoted by the numerals I and II. In region I $\sigma_1 < \sigma_2$, and in region II $\sigma_2 < \sigma_1$. Thus the constraining relation (1) may be rewritten as

$$\tau(x_i) = \varphi(x_i) \tag{3}$$

where

$$\varphi(x_i) \equiv 1/\sigma_1 \text{ in region I}$$

$$\equiv 1/\sigma_2 \text{ in region II} \tag{4}$$

It is noticed that $\varphi(x_i)$, which is continuous for all $x_i > 0$, has continuous partial derivatives in regions I and II but, in general, has a discontinuous normal derivative at the surface S. Equation (3) represents a constraining surface C, which is denoted by C_1 in region I and C_2 in region II. Geometrically, the minimum value of τ on C is being sought, and the question is, when does the minimum value of τ occur at the intersection of C and S?

Applying the method of Lagrange for constrained minima and introducing the undetermined multiplier λ, one may say that a necessary condition for a relative minimum of τ to exist at some point $\{\bar{x}_i\}$ in either region I or II is that the $n + 1$ equations†

$$(1 + \lambda)(\partial\tau/\partial x_i) - \lambda(\partial\varphi/\partial x_i) = 0 \qquad i = 1,2, \ldots n \tag{5}$$

$$\tau - \varphi = 0$$

be satisfied at $\{\bar{x}_i\}$. Without placing restrictions on the functions τ,σ_1, and σ_2, there is no reason why Eqs. (5) could not have a solution in either region I or II which gives the absolute minimum of τ. Only if solutions of Eqs. (5) do not give a relative minimum of τ at any point in either region I or II can it be said for certain that τ obtains its minimum on the surface S if, indeed, a minimum exists.

A set of sufficient conditions for solutions of Eqs. (5) to give no relative minimum of τ in region I or II is as follows:

1) τ is independent of one of the cross-section variables x_i, say x_k. Thus $\partial\tau/\partial x_k \equiv 0$.

2) σ_1 is either monotone increasing or decreasing with respect to x_k in region I, and σ_2 is either monotone increasing or decreasing with respect to x_k in region II. Thus $\partial\sigma_1/\partial x_k \neq 0$ in region I, and $\partial\sigma_2/\partial x_k \neq 0$ in region II. From Eqs. (4), this implies $\partial\varphi/\partial x_k \neq 0$ in regions I and II.

3) The function $\tau(x_i)$ does not obtain a relative minimum with respect to the unconstrained variables $x_i > 0$. This condition is satisfied for all physical thickness functions, since the effective thickness of a cross section always can be made smaller by taking certain dimensions smaller.

As a result of condition 1, the kth equation of Eqs. (5) becomes $\lambda\partial\varphi/\partial x_k = 0$. Because of condition 2, this reduces to $\lambda = 0$. Then the remaining members of Eqs. (5) reduce to

$$\partial\tau/\partial x_i = 0 \qquad i \neq k \tag{6a}$$

$$\tau - \varphi = 0 \tag{6b}$$

which are separated to display the fact that Eqs. (6a) now are uncoupled from Eq. (6b), since Eqs. (6a) do not involve x_k. Thus Eq. (6b) can be solved for x_k after Eqs. (6a) are solved. If Eqs. (6a) and (6b) have a solution $\{\bar{x}_i\}$ in either region I or II, that solution cannot give a (constrained) relative minimum of τ. For, by condition 3, at least one of the variables $x_i (i \neq k)$ may be varied to give a smaller value of τ, and because $\partial\varphi/\partial x_k \neq 0$ (cf., condition 2), Eq. (6b) still can be solved for x_k after the variation.‡

Thus it follows that conditions 1–3 are a sufficient set of conditions on the functions τ, σ_1, and σ_2 for the minimum of τ, if it exists, to occur on the surface S defined by $\sigma_1 = \sigma_2$.

Actually condition 2 can be sharpened to

(2′) $\text{sign}(\partial\sigma_1/\partial x_k \text{ in region I}) = -\text{sign}(\partial\sigma_2/\partial x_k \text{ in region II})$

That is, the monotone variations of σ_1 and σ_2 with respect to x_k in regions I and II, respectively, should have opposite sense. If they have the same sense, it follows from Eqs. (4) that φ is monotone increasing (or decreasing) with respect to x_k in both regions I and II. Then no minimum can exist on S. For, again, by condition 3, one of the variables, say x_l $(l \neq k)$, may be varied to reduce τ, and then x_k may be decreased (or increased) until the constraint $\tau = \varphi$ again is satisfied.

By simple examples, one can show that all of the conditions 1, 2′, and 3 are needed to insure that $\sigma_1 = \sigma_2$ is a neces-

† The question of Eqs. (5) having a solution on S is meaningless, since the vector $\{\partial\varphi/\partial x_i\}$ in general does not exist on S.

‡ This is a consequence of the following implicit function theorem: if $F(x,y) = 0$ is satisfied by a pair of values (x_0,y_0) so that $F(x_0,y_0) = 0$, then $F(x,y) = 0$ can be solved for y in terms of x in the neighborhood of x_0 if $\partial F/\partial y \neq 0$ in the neighborhood of (x_0,y_0).

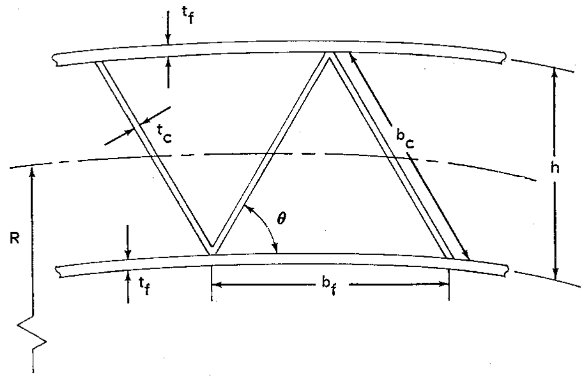

Fig. 1a Single truss-core section

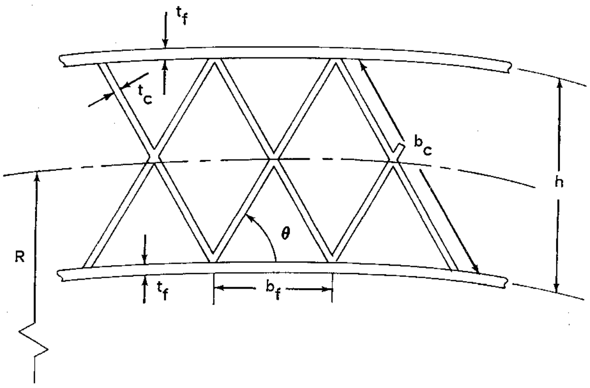

Fig. 1b Double truss-core section

sary condition for minimum weight. In practice, however, it is necessary only to check conditions 1 and 2′ since, as pointed out before, condition 3 is satisfied automatically in problems with physical meaning. In many problems of minimum weight design, conditions 1 and 2′ are satisfied when there are no additional constraints on the cross-sectional dimensions. For example, in the design of flat compression panels having unflanged integral stiffeners,[2] the stiffener pitch b serves as the variable x_k, and in the design of flat truss-core sandwich panels,[3] the sandwich depth h serves as x_k. Thus, determining the optimum cross-sectional dimensions in either of these cases with the use of the relation $\sigma_1 = \sigma_2$ is valid *as long as b or h is free to vary.* But if b or h is fixed a priori from other considerations, then the usual procedure no longer is valid, and the optimum design may have $\sigma_1 \neq \sigma_2$.

IV. Single and Double Truss-Core Sandwich Cylinders

The cylinders considered here have sandwich-type walls with cores made up of longitudinally running corrugations. Shown in Fig. 1 are idealized models, upon which the analysis is based, of the two types of composite walls considered. It is assumed that the sandwich faces are of equal thickness and that the elastic properties are the same for both core and face materials. The two modes of instability considered are elastic general buckling of the composite wall and elastic local buckling of the individual plate elements of the wall. In regard to the general buckling, it is assumed that "snap-through" buckling is suppressed by the relatively thick composite wall, so that linear stability theory is applicable.§

The critical stress σ_g for general buckling of corrugated-core sandwich cylinders with longitudinally running corrugations that are symmetrical, on the average, about the shell middle surface is given by[5]

$$\sigma_g = \pi^2 k_g (E/L^2)[I_f/(A_c + A_f)] \tag{7}$$

§ For corrugated-core sandwich cylinders, there is some experimental evidence to support this assumption.[4]

If the transverse shear deformation in the circumferential direction is not neglected,‖ this formula is cumbersome to apply, for then one does not obtain a closed expression for the buckling coefficient k_g. Instead, k_g is given as the minimum of a certain double sequence k_{pq} with respect to integral values of p and q. Because of the complexity of the expression for k_{pq}, one can obtain k_g only numerically after all the pertinent geometrical and physical properties of the cylinder are given. In order to avoid the difficulties associated with a numerical approach, an approximate analytical expression for k_g, valid for infinite transverse shear stiffness, is used, and then a reduction factor is applied to the resulting formula for σ_g to account for transverse shear deformation. The assumption of infinite transverse shear stiffness in the circumferential direction leads to the following approximate expressions[5] for k_g:

$$\begin{aligned} k_g &= \zeta + \frac{1}{1-\mu^2} + \frac{Z^2}{\pi^4\xi} \quad \text{for } \frac{Z^2}{\pi^4\xi} \leqslant \zeta + \frac{1}{1-\mu^2} \\ &= 2\left[\frac{Z^2}{\pi^4\xi}\left(\zeta + \frac{1}{1-\mu^2}\right)\right]^{1/2} \quad \text{for } \frac{Z^2}{\pi^4\xi} \geqslant \zeta + \frac{1}{1-\mu^2} \end{aligned} \tag{8}$$

where

$$\begin{aligned} \zeta &= I_c/I_f \\ \xi &= \frac{1 + (1-\mu^2)(A_c/A_f)}{1 + (A_c/A_f)} \\ Z^2 &= (2t_f/I_f)(L^4/R^2) \end{aligned} \tag{9}$$

For the truss-core cross sections, the face and core areas per unit width are given by

$$\begin{aligned} A_f &= 2t_f \\ A_c &= 2t_c b_c/b_f \end{aligned} \tag{10}$$

and the face and core moments of inertia per unit width are

$$\begin{aligned} I_f &= (t_f h^2/2)[1 + \tfrac{1}{3}(t_f/h)^2] \\ I_c &= t_c b_c h^2/6b_f \end{aligned} \tag{11}$$

Substituting Eqs. (10) and (11) into Eqs. (9) gives, for these cross sections,

$$\begin{aligned} \zeta &= \Lambda/3\beta \\ \xi &= [1 + (1-\mu^2)\Lambda]/(1+\Lambda) \\ Z^2 &= (4/\beta)(L^4/h^2R^2) \end{aligned} \tag{12}$$

where the notation

$$\begin{aligned} \Lambda &\equiv t_c b_c/t_f b_f \\ \beta &\equiv 1 + \tfrac{1}{3}(t_f/h)^2 \end{aligned} \tag{13}$$

is used. Since, for practical cases, $\Lambda \sim 1$, $L \sim R$, and $\beta \approx 1$, it follows from Eq. (12) that $\zeta \sim 1$, $\xi \sim 1$, and $Z^2 \sim (L/h)^2 \gg 1$. Hence, in cases of interest, the condition

$$Z^2/\pi^4\xi \geqslant \zeta + [1/(1-\mu^2)] \tag{14}$$

is satisfied so that only the second of Eqs. (8) need be used. Introducing a shear deformation reduction factor s and using Eqs. (10–12) and the second of Eqs. (8), one obtains from Eq. (7) the following expression for σ_g:

$$\sigma_g = s\,\frac{Eh}{R}\left\{\frac{\beta + \frac{1}{3}(1-\mu^2)\Lambda}{(1-\mu^2)(1+\Lambda)[1 + (1-\mu^2)\Lambda]}\right\}^{1/2} \tag{15}$$

The reduction factor s is a complicated function of all the pertinent geometrical and physical properties of the shell and can be obtained numerically for given shell properties as the ratio of the minimum value of k_{pq} for integral values of p and q to the value of k_g given by the second of Eqs. (8). In general, $s < 1$, and $s \to 1$ as the shear deformation becomes unimportant. Calculations for a representative corrugated-core sandwich indicate $s \approx 1$ for either very short $(L/R < 0.1)$

‖ For longitudinally running corrugations, the transverse shear deformation in the axial direction is negligible.

or very long ($L/R > 10$) cylinders and that s can take on values as low as 0.6 in the intermediate range.[5] In the development of optimum design charts, it is assumed that s is a slowly varying function of the cross-sectional variables x_i, so that $\partial s/\partial x_i$ can be neglected in the minimization process. Thus s is treated as a parameter that can be estimated for a given design. Since, as will become clear, the determination of an "optimum design" will depend on the assumed value of s, the true optimum design will be determined necessarily by a process of iteration with respect to the parameter s. The simplicity afforded by treating s as a parameter is achieved at the expense of some compromise of structural efficiency, i.e., the design obtained will be somewhat off optimum. However, in view of the idealized nature of the geometrical models of the sandwich cross sections, to do more than this is not warranted.

It is noted that Eq. (15) yields the critical stress for moderately long, thin-wall cylinders obtained from small-deflection theory, i.e., as the core thickness vanishes,

$$\Lambda \to 0 \qquad \beta \to \tfrac{4}{3}$$
$$t_f \to h \to t/2 \qquad s \to 1$$

and Eq. (15) gives

$$\sigma_g \to (Et/R)[1/3(1-\mu^2)]^{1/2}$$

a well-known result.[6] However, this value of critical stress is not valid for thin-wall cylinders, because finite deflection effects determine the critical load ("snap-through" buckling). Assuming, as mentioned earlier, that these effects are suppressed for the sandwich cylinders with $t_f/h \ll 1$ (or $\beta \approx 1$), one may take Eq. (15), with β replaced by unity, to be valid when condition (14) is satisfied. Substituting Eqs. (12), one may rewrite condition (14) as

$$\frac{L^2}{hR} \geqslant \frac{\pi^2}{2}\left\{\frac{[1+(1-\mu^2)\Lambda][\beta+\frac{1}{3}(1-\mu^2)\Lambda]}{(1-\mu^2)(1+\Lambda)}\right\}^{1/2}$$

The local instability of flat plate elements has the critical stress given by[7]

$$\sigma_l = [\pi^2/12(1-\mu^2)]k_l\,E(t_f/b_f)^2 \tag{16}$$

where, for either a single or double truss core, k_l is given graphically in Ref. 7 in the form

$$k_l = k_l[(t_c/t_f),\theta]$$

To apply the theory given in Sec. III, one must choose a set of independent geometric variables x_i which defines the sandwich cross section and express the critical stress σ_g for general buckling, the critical stress σ_l for local buckling, and the effective thickness (cross-sectional area per unit of circumferential width) $\bar{t}$ in terms of the chosen x_i. For the truss-core cross sections there are four independent geometric variables, which may be taken as θ, h, t_f, and t_c/t_f. From Eqs. (15) and (16), the general buckling stress σ_g and the local buckling stress σ_l are the following functions of these variables:

$$\sigma_g = \frac{E}{(1-\mu^2)^{1/2}}\frac{s}{R}h\left\{\frac{1+\frac{1}{3}(1-\mu^2)\Lambda}{(1+\Lambda)[1+(1-\mu^2)\Lambda]}\right\}^{1/2}$$
$$\sigma_l = \frac{m^2\pi^2}{48}\frac{E}{1-\mu^2}\frac{t_f^2}{h^2}k_l\left(\frac{t_c}{t_f},\theta\right)\tan^2\theta \tag{17}$$

where, from (13),

$$\Lambda = (m/2)(t_c/t_f)\sec\theta \tag{18}$$

in which $m = 1$ for a single truss core and $m = 2$ for a double truss core. A simple calculation shows the effective thickness $\bar{t}$ to be given by

$$\bar{t} = 2(1+\Lambda)t_f \tag{19}$$

The problem then is to minimize $\bar{t}$ with respect to θ, h, t_f, and t_c/t_f subject to the constraint

$$\bar{t} = q/\min(\sigma_l,\sigma_g) \tag{20}$$

A. *h* Free

With h free, regardless of constraint on any of the three remaining variables θ, t_f, or t_c/t_f, conditions 1, 2′, and 3 are satisfied, viz:

1) $\bar{t}$ is independent of h.

2) σ_g is monotone increasing with respect to h, and σ_l is monotone decreasing with respect to h.

3) $\bar{t}$ does not obtain a relative minimum with respect to θ, t_f, and t_c/t_f.

As a consequence, for h free, $\bar{t}$ obtains its minimum value for $\sigma_l = \sigma_g$. This condition simplifies the constraining relation (20), and together they may be rewritten, with the aid of Eqs. (17) and (19), as

$$\frac{m^2\pi^2E}{48(1-\mu^2)}\frac{t_f^2}{h^2}k_l\tan^2\theta = \frac{Es}{R(1-\mu^2)^{1/2}}h\left\{\frac{1+\frac{1}{3}(1-\mu^2)\Lambda}{(1+\Lambda)[1+(1-\mu^2)\Lambda]}\right\}^{1/2} = \frac{q}{2t_f(1+\Lambda)} \tag{21}$$

Equations (21) are two simultaneous equations among the four cross-sectional variables. Solving for t_f and h in terms of θ and t_c/t_f and putting in dimensionless form gives

$$\tau_f \equiv t_f\left(\frac{E}{q}\right)^{3/5}\left(\frac{s}{R}\right)^{2/5} = \left\{\frac{6(1-\mu^2)^2}{m^2\pi^2}\frac{1+(1-\mu^2)\Lambda}{(1+\Lambda)^2[1+\frac{1}{3}(1-\mu^2)\Lambda]k_l\tan^2\theta}\right\}^{1/5} \tag{22}$$

$$\eta \equiv h(E/q)^{2/5}(s/R)^{3/5} = \psi/\tau_f \tag{23}$$

where

$$\psi(\Lambda) = \frac{1}{2}\left\{\frac{(1-\mu^2)[1+(1-\mu^2)\Lambda]}{(1+\Lambda)[1+\frac{1}{3}(1-\mu^2)\Lambda]}\right\}^{1/2} \tag{24}$$

Finally, from Eq. (19), the dimensionless effective thickness $\bar{\tau}$ is given in terms of θ and t_c/t_f by

$$\bar{\tau} \equiv \bar{t}(E/q)^{3/5}(s/R)^{2/5} = 2(1+\Lambda)\tau_f \tag{25}$$

Equations (22–25) were used, with $\mu = \frac{1}{3}$, to construct the τ_f, η, and $\bar{\tau}$ contour maps (Figs. 2 and 3).

It may be shown from Eqs. (23) and (24) that intersections of τ_f and η curves can occur only if the product $\eta\tau_f = \psi < 2^{1/2}/3$, since, for $\psi \geqslant 2^{1/2}/3$, Eq. (24) yields no positive solution for Λ.

B. *h* Fixed

If h (or η) is fixed by some consideration other than compressive load-carrying ability, then reference to Eqs. (17)

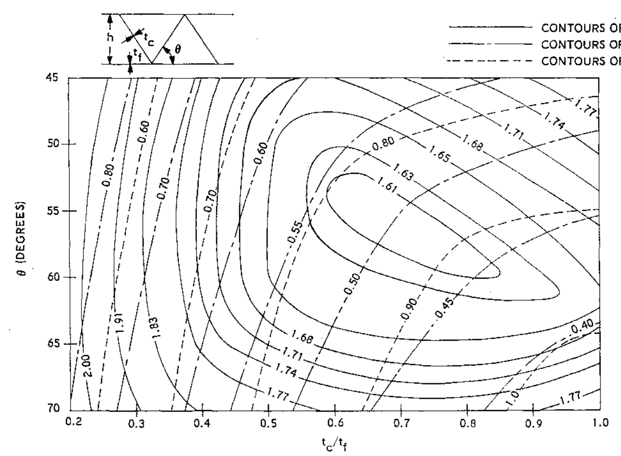

Fig. 2 Single truss core

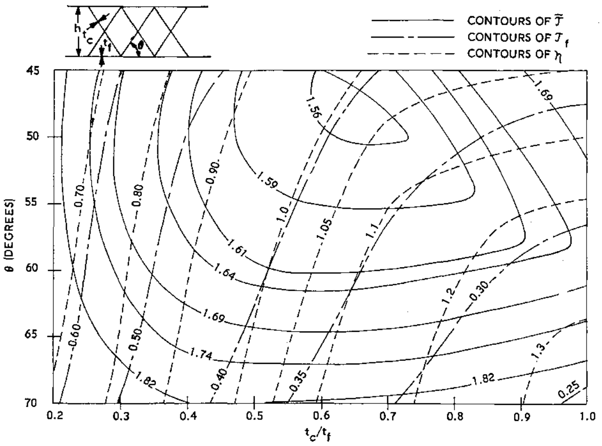

Fig. 3 Double truss core

and (19) shows that neither of the conditions 1 and 2′ is satisfied, viz:

1) $\bar{t}$ is not independent of θ, t_f, or t_c/t_f.

2) Although σ_l is monotone increasing with respect to t_f, σ_g is independent of t_f, and the variation of σ_l and σ_g with respect to θ or t_c/t_f is not apparently monotone.

As a consequence, if h is fixed one cannot, a priori, assume that $\bar{t}$ obtains its minimum value with respect to the variables t_f, θ, and t_c/t_f for $\sigma_l = \sigma_g$, and, in fact, for a particular value of η, $\bar{t}$ may obtain its minimum for $\sigma_l \neq \sigma_g$. In this case, a more general procedure is needed to find the optimum design.

The equation of the surface S, obtained by setting $\sigma_l = \sigma_g$ from Eqs. (17) and solving for t_f is, in terms of dimensionless variables,

$$\tau_f = \frac{4(3)^{1/2}(1-\mu^2)^{1/4}}{m\pi}\,\eta^{3/2}\,\frac{1}{k_l^{1/2}\tan\theta} \times \left\{\frac{1+\frac{1}{3}(1-\mu^2)\Lambda}{(1+\Lambda)[1+(1-\mu^2)\Lambda]}\right\}^{1/4} \qquad (26)$$

S divides the (θ, τ_f, t_c/t_f) space into two regions: region L, where $\sigma_l < \sigma_g$; and region G, where $\sigma_g < \sigma_l$. Points in region L satisfy the inequality

$$\tau_f < \frac{4(3)^{1/2}(1-\mu^2)^{1/4}}{m\pi}\,\eta^{3/2}\,\frac{1}{k_l^{1/2}\tan\theta} \times \left\{\frac{1+\frac{1}{3}(1-\mu^2)\Lambda}{(1+\Lambda)[1+(1-\mu^2)\Lambda]}\right\}^{1/4} \qquad (27a)$$

and points in region G satisfy

$$\tau_f > \frac{4(3)^{1/2}(1-\mu^2)^{1/4}}{m\pi}\,\eta^{3/2}\,\frac{1}{k_l^{1/2}\tan\theta} \times \left\{\frac{1+\frac{1}{3}(1-\mu^2)\Lambda}{(1+\Lambda)[1+(1-\mu^2)\Lambda]}\right\}^{1/4} \qquad (27b)$$

Using Eqs. (17) and (19), the constraining relation (20) may be solved for τ_f to yield in region L

$$\tau_f = 2\left[\frac{3(1-\mu^2)}{m^2\pi^2}\right]^{1/3}\eta^{2/3}\,\frac{1}{[(1+\Lambda)k_l\tan^2\theta]^{1/3}} \qquad (28a)$$

and in region G

$$\tau_f = \frac{(1-\mu^2)^{1/2}}{2\eta}\left\{\frac{1+(1-\mu^2)\Lambda}{(1+\Lambda)[1+\frac{1}{3}(1-\mu^2)\Lambda]}\right\}^{1/2} \qquad (28b)$$

Substituting Eqs. (28) into Eq. (25) and inequalities (27) eliminates the variable τ_f in accordance with the constraint (20). The results are as follows.

If

$$\frac{(1+\Lambda)^{1/4}}{k_l^{1/2}\tan\theta}\left[\frac{1+\frac{1}{3}(1-\mu^2)\Lambda}{1+(1-\mu^2)\Lambda}\right]^{3/4} > \frac{m\pi(1-\mu^2)^{1/4}}{8(3)^{1/2}\eta^{5/2}} \quad \text{(region } L\text{)}$$

then

$$\bar{\tau} = 4\left[\frac{3(1-\mu^2)}{m^2\pi^2}\right]^{1/3}\eta^{2/3}\left[\frac{(1+\Lambda)^2}{k_l\tan^2\theta}\right]^{1/3} \qquad (29a)$$

If

$$\frac{(1+\Lambda)^{1/4}}{k_l^{1/2}\tan\theta}\left[\frac{1+\frac{1}{3}(1-\mu^2)\Lambda}{1+(1-\mu^2)\Lambda}\right]^{3/4} < \frac{m\pi(1-\mu^2)^{1/4}}{8(3)^{1/2}\eta^{5/2}} \quad \text{(region } G\text{)}$$

then

$$\bar{\tau} = \frac{(1-\mu^2)^{1/2}}{\eta}\left\{\frac{(1+\Lambda)[1+(1-\mu^2)\Lambda]}{1+\frac{1}{3}(1-\mu^2)\Lambda}\right\}^{1/2} \qquad (29b)$$

Thus, given η, Eqs. (29) give $\bar{\tau} = \bar{\tau}(\theta, t_c/t_f)$, which then is to be minimized with respect to the remaining variables θ and t_c/t_f. Since k_l $(\theta, t_c/t_f)$ is a numerical function, this is necessarily a numerical process. For design purposes, for each value of η a pair of contour maps, similar to Figs. 2 and 3, giving lines of constant τ_f and $\bar{\tau}$, could be constructed from Eqs. (28) and (29). However, this work has not been performed.

V. Conclusion

In conclusion, it can be stated that the usual procedure of equating critical stresses in seeking optimum designs of compression structures with two instability modes is valid in many problems where there are no constraints placed on the design. It has been shown, however, that a particular geometric variable of the structure plays a special role in insuring the validity of this procedure. If this particular variable is constrained off its optimum value, then the forementioned procedure may lead to a nonoptimum design. That optimum constrained designs for which the critical stresses are unequal actually do occur has been verified for some simple cases (e.g., corrugated wide columns), although such calculations have not been presented here.

In the case of truss-core sandwich cylinders subject to general and local buckling of the wall, the sandwich depth plays this special role. The analysis presented in the paper shows that, if the sandwich depth is free, the condition of equality of the critical stresses is a necessary condition for minimum weight and hence may be used in seeking optimum designs. The graphical results presented for this case allow rapid determination of minimum-weight designs for single and double truss-core sandwich cylinders. It is noted that, if transverse shear deformation is neglected, the double truss core shows a theoretical weight saving of about 3% over the single truss core for unconstrained designs. The equations necessary for constructing similar charts for each case of given sandwich depth have been presented, but the associated numerical work has not been performed.

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